MID-SEMESTER EXAM — ISI BANGALORE — FEBRUARY 2023

DIFFERENTIAL TOPOLOGY — 100 POINTS

Notes.

(a) You may freely use any result proved in class or in the textbook unless you have been asked to prove the same. Use your judgement. All other steps must be justified.

(b) There are a total of **105** points in the paper. You will be awarded a maximum of **100**.

(c) $\mathbb{R}$ = real numbers.

(d) All the manifolds will be assumed to be submanifolds in some Euclidean space $\mathbb{R}^N$.

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1. [10 points] Let X and Y be manifolds and let $f\colon X \to Y$ be a smooth map.

   (i) Show that the subset Γ_f of $X \times Y$ defined by
   $$\Gamma_f = \text{graph}(f) = \{(x, f(x)) \mid x \in X\}$$
   is diffeomorphic to X and hence a submanifold of $X \times Y$.

   (ii) Prove that the tangent space to Γ_f at $(x, f(x))$ is the graph of $df_x\colon T_x(X) \to T_{f(x)}(Y)$, i.e.,
   $$T_{(x,f(x))}(\Gamma_f) = \{(v, df_x(v)) \mid v \in T_x(X)\}.$$

2. [10 points] Give an example of a one-one immersion $f\colon X \to Y$ of manifolds which is not proper.

3. [15 points] Let X be a k-dimensional submanifold of $\mathbb{R}^N$. For any $p \in X$, prove that there is an open neighbourhood $V \subset X$ of p and an open set $U \subset \mathbb{R}^k$ together with a smooth map $g = (g_1, \ldots, g_{N-k})\colon U \to \mathbb{R}^k$ such that upto a permutation of coordinate indices, V is the graph of g.

4. [10 points] Compute the critical points and critical values of the function $f(x, y) = (x, xy)$. Write $\mathbb{R}^2$ as a disjoint union $\mathbb{R}^2 = S_1 \cup S_2 \cup S_3$, where S_1 is the set of critical values of f, S_2 the set of regular values in the image of f and S_3 the set of elements not lying in the image of f.

5. [20 points] Let $SL(n)$ denote the multiplicative group of $n \times n$ matrices over $\mathbb{R}$ with determinant 1. Prove that $SL(n)$ is a submanifold of $M(n)$ (the set of all $n\times n$ matrices over $\mathbb{R}$) whose tangent space at the identity matrix I_n is the set of all matrices in $T_{I_n}(M(n)) = M(n)$ having trace equal to zero.

6. [20 points] Let $f\colon X \to Y$ be a smooth map of manifolds and let $Z \subset Y$ be a submanifold. Define what it means for f to be transversal to Z. When that happens, prove that $f^{-1}Z$ is a submanifold of X whose codimension in X equals that of Z in Y.

7. [20 points] Let D denote the open unit disc in $\mathbb{R}^2$, i.e., $D = \{p \in \mathbb{R}^2 \mid \|p\| < 1\}$ and let $\overline{D}$ denote its closure in $\mathbb{R}^2$. In each of the following cases, prove or disprove that there is a manifold X and a smooth map $f\colon X \to \mathbb{R}^2$ such that $f(X)$ and the set A of critical values of f are as specified.

(i) $f(X) = D$, $A = \{(0,0)\}$.
(ii) $f(X) = \overline{D}$, $A = \{(0,0)\}$.
(iii) $f(X) = \overline{D}$, $A = \{p \in \overline{D} \mid \|p\| = 1\}$.
(iv) $f(X) = D$, $A = \{p \in D \mid \|p\| > 1/2\}$.